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1975 J. Phys. A: Math. Gen. 8 1975

(http://iopscience.iop.org/0305-4470/8/12/013)

J. Phys. A: Math. Gen., Vol. 8, No. 12, 1975. Printed in Great Britain. © 1975

Momentum and angular momentum for some exact solutions of Fokker's electrodynamics

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Received 18 December 1974, in final form 7 July 1975

Abstract. Four-momentum and angular four-momentum are calculated and discussed for the circular non-planar solutions of Fokker's electrodynamics, recently found by Chern and Havas. The initial value problem in this electrodynamics is also considered taking into account the fact that it is possible to construct different circular trajectories with the same initial positions and velocities.

1. Introduction

For some time now, the difficulties which spoil the applications of field theories in electrodynamics have been recognized. As a consequence of this fact, a movement started‡ for the re-examination of concepts, such as field, causality, etc.

This paper is mainly concerned with Fokker's§ electrodynamics (Fokker 1929), an action-at-a-distance theory that describes relativistic systems of interacting charges through an action principle. Fokker's electrodynamics was later completed by Wheeler and Feynman (1945, 1949). The difference being that their theory of radiation uses the condition of 'complete absorption' (a condition which we do not make any use of in this paper).

Fokker started from the following Poincaré-invariant action integral (Fokker 1929)||

$$S = -\sum_a m_a \int [-(\dot{z}_a\dot{z}_a)]^{1/2}\,\mathrm{d}\tau_a + \sum\sum_{a<a'} e_a e_{a'} \int\int (\dot{z}_a\dot{z}_{a'})\delta[(z_a - z_{a'})^2]\,\mathrm{d}\tau_a\,\mathrm{d}\tau_{a'}. \tag{1.1}$$

The equations of motion which are obtained from (1.1) have a peculiarity: they are differential-difference equations in which the difference depends on time (this represents the fact that the interaction is not propagated with infinite velocity). No general theory exists for such equations and for this reason the search for exact solutions is not a trivial one. In fact, up to now only Smith (1960), Schild (1963) and Chern and Havas (1973)

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‡ A careful discussion and earlier references on the subject can be found in Wheeler and Feynman (1945, 1949).

§ To be more correct we should speak of Schwarzschild–Tetrode–Fokker's electrodynamics (Schwarzschild 1903, Tetrode 1922, Fokker 1929). For the sake of economy however, we shall mention only Fokker.

|| We shall take the Minkowski metric with signature $+2$ and units will be chosen so that the speed of light will be $c = 1$. $a, a' = 1, 2$, $a \neq a'$; $\alpha, \beta, \ldots = 0, 1, 2, 3$; $i, j, \ldots = 1, 2, 3$. The following notation will be used: $(xy) \equiv x^\alpha y_\alpha$, $x^2 \equiv (xx)$.

have found exact solutions of a very particular type: flat (ie the configuration is such that for every instant there exists a plane containing the n bodies) and circular. More recently, Chern and Havas (1974) have been able to show that there exists, under certain conditions, an infinite number of exact solutions.

The study of the properties of these solutions is of great importance because one question remaining to be answered is that of their physical meaning (stability, etc). In this sense, we calculate here the four-momentum and the angular four-momentum for some two-body non-planar exact solutions of Chern and Havas (1973).

We also analyse the initial value problem in Fokker's electrodynamics taking into account the fact that it is possible to construct different circular trajectories with the same initial positions and velocities.

2. The non-planar solutions of Chern and Havas

As is well known in Fokker's electrodynamics (considering, as in our case, the two-body problem) the four-acceleration of a particle $\underline{a}$ is

$$\xi^{\alpha}_{a} = \tfrac{1}{2}(\xi^{\alpha}_{a+} + \xi^{\alpha}_{a-}) \tag{2.1}$$

where

$$\xi^{\alpha}_{a\epsilon} = \epsilon g m_a^{-1} (\hat{z}_{\underline{a}'\epsilon} \hat{\dot{z}}_{\underline{a}'\epsilon})^{-2} \{(\hat{\xi}_{a'\epsilon} \dot{z}_a) \hat{z}^{\alpha}_{a'\epsilon} - (\hat{z}_{\underline{a}'\epsilon} \dot{z}_a) \hat{\xi}_{a'\epsilon} + (\hat{z}_{\underline{a}'\epsilon} \hat{\dot{z}}_{a'\epsilon})^{-1} [1 + (\hat{z}_{\underline{a}'\epsilon} \hat{\xi}_{a'\epsilon})] \cdot [(\hat{z}_{\underline{a}'\epsilon} \dot{z}_a) \hat{\dot{z}}^{\alpha}_{a'\epsilon} - (\dot{z}_a \hat{\dot{z}}_{a'\epsilon}) \hat{z}^{\alpha}_{\underline{a}'\epsilon}]\},$$

with $\epsilon = \pm 1$; $g \equiv e_a e_{a'}$ (e_a are the charges of the particles); $\hat{z}^{\alpha}_{\underline{a}'\epsilon} \equiv z^{\alpha}_a - \hat{z}^{\alpha}_{a'\epsilon}$, $\hat{z}^{\alpha}_{a'\epsilon}$ being the intersection of the light cone emanating from the point z^{α}_a with the trajectory of particle $\underline{a}'$; $\hat{\dot{z}}^{\alpha}_{a'\epsilon}$ is the unit tangent vector to this trajectory at the point $\hat{z}^{\alpha}_{a'\epsilon}$ and $\hat{\xi}^{\alpha}_{a'\epsilon}$ the four-acceleration of $\underline{a}'$ at the same point (see figure 1).

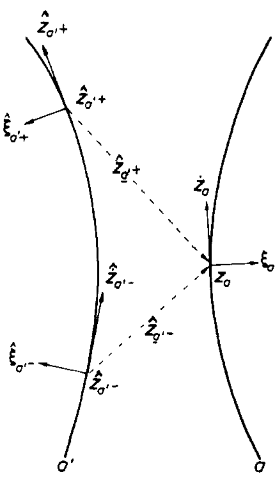

Figure 1.

We shall now consider the circular non-planar two-body solutions of Fokker's electrodynamics which have recently been obtained by Chern and Havas (1973).

The position of particle $\underline{a}$ is given by

$$z_a^\alpha = (t_a; \eta_a r_a \cos \omega t_a, \eta_a r_a \sin \omega t_a, h_a),$$

where two different possibilities will be considered: $\eta_a = +1$, $\eta_{a'} = \pm 1$, and where r_a, ω, $h_a (h_a \neq h_{a'})$ are positive constants.

The following relations are satisfied:

$$v_a = r_a \omega, \qquad \frac{\mathrm{d}\phi_a}{\mathrm{d}\tau_a} = \omega\gamma_a$$

$(\phi_a \equiv \omega t_a, v_a^2 \equiv v_{ai} v_a^i, v_a^i \equiv \mathrm{d}z_a^i/\mathrm{d}t_a, \gamma_a \equiv (1-v_a^2)^{-1/2})$.

In that which follows, we shall use ϕ_a as the parameter instead of t_a. Then we have

$$z_a^\alpha = \omega^{-1}(\phi_a; \eta_a v_a \cos \phi_a, \eta_a v_a \sin \phi_a, b_a), \qquad (h_a \equiv \omega^{-1} b_a) \tag{2.2a}$$

$$\dot{z}_a^\alpha = \gamma_a (1; -\eta_a v_a \sin \phi_a, \eta_a v_a \cos \phi_a, 0), \qquad (\dot{z}_a \equiv \mathrm{d}z_a^\alpha/\mathrm{d}\tau_a; \dot{z}_a^\alpha \dot{z}_{a\alpha} = -1) \tag{2.2b}$$

$$\xi_a^\alpha = -\eta_a \omega \gamma_a^2 v_a (0; \cos \phi_a, \sin \phi_a, 0), \qquad (\xi_a^\alpha \equiv \mathrm{d}\dot{z}_a^\alpha/\mathrm{d}\tau_a). \tag{2.2c}$$

Given $z_a^\alpha(\phi_a)$, $\hat{z}_{a'\epsilon}^\alpha$ are defined by

$$(\hat{z}_{\underline{a}'\epsilon} \hat{z}_{\underline{a}'\epsilon}) = 0, \qquad \operatorname{sgn} z_{\underline{a}'\epsilon}^0 = -\epsilon,$$

so that we can write (in that which follows, we shall substitute ϕ_a by ϕ):

$$\hat{z}_{a'\epsilon}^\alpha = \omega^{-1}(\phi + \epsilon\theta; \eta_{a'} v_{a'} \cos(\phi + \epsilon\theta), \eta_{a'} v_{a'} \sin(\phi + \epsilon\theta), b_{a'}), \tag{2.3a}$$

$$\hat{z}_{\underline{a}'\epsilon}^\alpha = \omega^{-1}(-\epsilon\theta; \eta_a v_a \cos \phi - \eta_{a'} v_{a'} \cos(\phi + \epsilon\theta), \eta_a v_a \sin \phi - \eta_{a'} v_{a'} \sin(\phi + \epsilon\theta), b),$$
$$(b \equiv b_a - b_{a'}), \tag{2.3b}$$

$$\hat{\dot{z}}_{a'\epsilon}^\alpha \equiv \gamma_{a'}(1; -\eta_{a'} v_{a'} \sin(\phi + \epsilon\theta), \eta_{a'} v_{a'} \cos(\phi + \epsilon\theta), 0), \tag{2.3c}$$

$$\hat{\xi}_{a'\epsilon}^\alpha = -\eta_{a'} \omega \gamma_{a'}^2 v_{a'} (0; \cos(\phi + \epsilon\theta), \sin(\phi + \epsilon\theta), 0), \tag{2.3d}$$

where θ stands for the retardation angle, that is, for the positive root of

$$\theta^2 - v_a^2 - v_{a'}^2 + 2\chi v_a v_{a'} \cos \theta - b^2 = 0, \qquad (\chi \equiv \eta_a \eta_{a'}). \tag{2.4}$$

Substituting expressions (2.2) and (2.3) in (2.1), after a straightforward calculation we get

$$m_a \omega^{-1} \gamma_a v_a v_{a'}^{-1} (\theta - \chi v_a v_{a'} \sin \theta)^2 = \chi g(\sin \theta - \theta \cos \theta) \tag{2.5}$$

$$v_a^2 v_{a'}^2 - \chi v_a v_{a'} (2 \cos \theta + \theta \sin \theta) + 1 = 0 \tag{2.6}$$

which, together with (2.4), allow the equations of motion to be identically satisfied.

From equations (2.5) the following can be obtained:

$$m_a \gamma_a v_a^2 = m_{a'} \gamma_{a'} v_{a'}^2, \tag{2.7}$$

which express—as we shall see later—the fact that, due to the symmetry of the situation, both particles contribute the same amount to the mass defect.

It is interesting to note that the conditions (2.4), (2.5) and (2.6) do not prohibit the existence of solutions with $g > 0$ or with $g < 0$. This does not happen with Smith and Schild's planar solution (Smith 1960, Schild 1963), where only the possibility $g < 0$ is allowed.

3. On the well-posed initial value problem in Fokker's electrodynamics

A special case of the solutions we are studying is the one for which $\chi = +1$, $v_1 = v_2$. It was pointed out by Chern and Havas (1973) that this solution exhibits a peculiar property: given the solution, it is possible to construct, with the same initial positions and velocities, another solution which also belongs to this class ($\chi = +1$, $v_1 = v_2$); this can be accomplished by means of a mirror transformation. After their 1973 paper Chern and Havas (1974) have found that there exist either infinitely many or at least two exact solutions with the same initial positions and velocities, depending on the masses of the particles and on conditions such as suitably given initial separation and appropriate initial velocities. Obviously, this fact clashes with the initial values situation in most classical theories. In a recent paper, Murdock (1974) has pointed out that if the equations of motion are differential-difference equations of the same type that appear in Fokker's electrodynamics then the precise specifications of a well-posed initial value two-body problem can be set up, so that to a given set of initial values corresponds a unique solution. To do this, Murdock needs, as initial values, arcs of trajectories verifying certain conditions in order not to violate causality†. With this in mind, it is not possible to construct more than one of these solutions ($\chi = \pm 1$, $v_1 = v_2$ or $v_1 \neq v_2$) corresponding to the same initial values.

In fact, applying Murdock's analysis we have determined a possible pair of arcs needed in order to determine uniquely the solution corresponding to $\chi = +1$, $v_1 = v_2$. The parametric equations obtained are

$$\gamma_a^\alpha(u) = \omega^{-1}(u; v\cos u, v\sin u, b_a), \qquad u \in [-\theta, \theta]$$

where θ is the same retardation angle we have been considering.

Since θ verifies (2.6) it is not difficult to demonstrate that

$$\theta = 2n\pi + \psi_n, \qquad \psi_n \in (0, l_n), \qquad l_n < \pi, \qquad n = 1, 2, \ldots .$$

The meaning of these solutions is that the arcs needed are at least twice the circular orbits. Of course, this fact does not represent anything special in M_4, but nevertheless, it is quite surprising for an R_3-mind!

The same analysis can be done for the remaining solutions of Chern and Havas that we are considering here. For $\chi = \pm 1, (v_a, v_{a'})$, the parametric equations of one possible pair of arcs are

$$\tilde{\gamma}_a^\alpha(u) = \omega^{-1}(u; \eta_a v_a \cos u, \eta_a v_a \sin u, b_a) \qquad u \in [-\theta, \theta]$$

where

$$\chi = +1, \theta = 2n\pi + \Omega_n, \Omega_n \in (0, p_n), p_n < \pi \qquad n = 1, 2, \ldots$$

$$\chi = -1, \theta = (2n+1)\pi + \Lambda_n, \Lambda_n \in (0, q_n), q_n < \pi \qquad n = 1, 2, \ldots .$$

In the case of Smith and Schild's planar solution (Smith 1960, Schild 1963), Murdock's analysis leads to

$$\bar{\gamma}_a^\alpha(u) = \omega^{-1}(u; \eta_a v_a \cos u, \eta_a v_a \sin u, 0), \qquad u \in [-\theta, \theta]$$

$$\theta \in (0, 1{\cdot}4782), \qquad \eta_a = 1, \qquad \eta_{a'} = -1.$$

† The non-violation of causality is understood as follows: no point of both arcs of trajectories is simultaneously in the domains of past and future influence of the other.

4. Momentum and angular momentum

In Fokker's electrodynamics, the four-momentum and the angular four-momentum concerning the origin of a system of two point charges are (Wheeler and Feynman 1949)

$$
\begin{aligned}
P^{\mu} =\ & m_a u_a^{\mu} + m_{a'} u_{a'}^{\mu} + e_a A_{a'}^{\mu}(x_a^{\lambda}) + e_{a'} A_a^{\mu}(x_{a'}^{\lambda}) \\
& + 2g\left(\int_0^{+\infty} \mathrm{d}\tau_a \int_{-\infty}^{0} \mathrm{d}\tau_{a'} - \int_{-\infty}^{0} \mathrm{d}\tau_a \int_0^{+\infty} \mathrm{d}\tau_{a'}\right) \delta'(z_{aa'}^2) z_{aa'}^{\mu} (\dot z_a \dot z_{a'}), \qquad (4.1)
\end{aligned}
$$

$$
\begin{aligned}
L^{\mu\nu} =\ & x_a^{\mu}(m_a u_a^{\nu} + e_a A_{a'}^{\nu}(x_a^{\lambda})) - x_a^{\nu}(m_a u_a^{\mu} + e_a A_{a'}^{\mu}(x_a^{\lambda})) + x_{a'}^{\mu}(m_{a'} u_{a'}^{\nu} + e_{a'} A_a^{\nu}(x_{a'}^{\lambda})) \\
& - x_{a'}^{\nu}(m_{a'} u_{a'}^{\mu} + e_{a'} A_a^{\mu}(x_{a'}^{\lambda})) + 2g\left(\int_0^{+\infty} \mathrm{d}\tau_a \int_{-\infty}^{0} \mathrm{d}\tau_{a'} - \int_{-\infty}^{0} \mathrm{d}\tau_a \int_0^{+\infty} \mathrm{d}\tau_{a'}\right) \\
& \times [\tfrac{1}{2}\delta(z_{aa'}^2)(\dot z_a^{\mu} \dot z_{a'}^{\nu} - \dot z_a^{\nu} \dot z_{a'}^{\mu}) - \delta'(z_{aa'}^2)(z_a^{\mu} z_{a'}^{\nu} - z_a^{\nu} z_{a'}^{\mu})(\dot z_a \dot z_{a'})] \qquad (4.2)
\end{aligned}
$$

where $A_a^{\mu}(x_{a'}^{\lambda})$ is the half-retarded plus half-advanced Lienard–Wiechert potential produced by charge $\underline{a}$ at $x_{a'}^{\lambda}$ given by

$$
A_a^{\mu}(x_{a'}^{\lambda}) = e_a \int_{-\infty}^{+\infty} \mathrm{d}\tau_a\, \delta[(x_{a'} - z_a)^2] \dot z_a^{\mu},
$$

$z_{aa'} \equiv z_a - z_{a'}$, and $x_a^{\alpha} \equiv z_a^{\alpha}(0)$, $u_a^{\alpha} \equiv \dot z_a^{\alpha}(0)$.

It is not difficult to see that (4.1)–(4.3) are equivalent to

$$
P^{\mu} = m_a u_a^{\mu} + m_{a'} u_{a'}^{\mu} + e_a A_{a'}(x_a^{\lambda}) + e_{a'} A_a(x_{a'}^{\lambda}) + \tfrac{1}{2} g(B^{\mu} + C^{\mu})
$$

where

$$
B^{\mu} \equiv -\sum_{\epsilon} \frac{\hat x_{a\epsilon}^{\mu}(\hat u_{a\epsilon} u_{a'})}{|(\hat u_{a\epsilon} \hat x_{\underline{a}\epsilon})|(u_{a'} \hat x_{\underline{a}\epsilon})}
$$

$$
\begin{aligned}
C^{\mu} \equiv\ & \omega^{-1}\gamma_a^{-1} \sum_{\epsilon} \int_0^{\epsilon\theta} \mathrm{d}\phi |(\check z_{\underline{a}'\epsilon} \check{\dot z}_{a'\epsilon})|^{-3} [\![\{(\check z_{\underline{a}'\epsilon} \check{\dot z}_{a'\epsilon})(\dot z_a \check\xi_{a'\epsilon}) - [1 + (\check z_{\underline{a}'\epsilon} \check\xi_{a'\epsilon})](\dot z_a \check{\dot z}_{a'\epsilon})\} \check z_{\underline{a}'\epsilon}^{\mu} \\
& - (\dot z_a \check{\dot z}_{a'\epsilon})(\check z_{\underline{a}'\epsilon} \check{\dot z}_{a'\epsilon}) \check{\dot z}_{a'\epsilon}^{\mu}]\!]
\end{aligned}
$$

being $\check w_{a'\epsilon}^{\alpha} \equiv \hat w_{a'(-\epsilon)}^{\alpha}$;

$$
\begin{aligned}
L^{\mu\nu} =\ & x_a^{\mu}(m_a u_a^{\nu} + e_a A_{a'}^{\nu}(x_a^{\lambda})) - x_a^{\nu}(m_a u_a^{\mu} + e_a A_{a'}^{\mu}(x_a^{\lambda})) + x_{a'}^{\mu}(m_{a'} u_{a'}^{\nu} + e_{a'} A_a^{\nu}(x_{a'}^{\lambda})) \\
& - x_{a'}^{\nu}(m_{a'} u_{a'}^{\mu} + e_{a'} A_a^{\mu}(x_{a'}^{\lambda})) + \tfrac{1}{2} g(B^{\mu\nu} + C^{\mu\nu}),
\end{aligned}
$$

where

$$
B^{\mu\nu} \equiv -\sum_{\epsilon} \frac{(\hat x_{a\epsilon}^{\mu} x_{a'}^{\nu} - \hat x_{a\epsilon}^{\nu} x_{a'}^{\mu})(\hat u_{a\epsilon} u_{a'})}{|(\hat u_{a\epsilon} \hat x_{\underline{a}\epsilon})|(u_{a'} \hat x_{\underline{a}\epsilon})}
$$

$$
\begin{aligned}
C^{\mu\nu} \equiv\ & \omega^{-1}\gamma_a^{-1} \int_0^{\epsilon\theta} \mathrm{d}\phi |(\check{\dot z}_{a'} \check z_{\underline{a}'\epsilon})|^{-1} (\dot z_a^{\mu} \check{\dot z}_{a'\epsilon}^{\nu} - \dot z_a^{\nu} \check{\dot z}_{a'\epsilon}^{\mu}) - \omega^{-1}\gamma_a^{-1} \int_0^{\epsilon\theta} \mathrm{d}\phi |(\check{\dot z}_{a'\epsilon} \check z_{\underline{a}'\epsilon})|^{-3} \\
& \times \{(\check{\dot z}_{a'\epsilon} \check z_{\underline{a}'\epsilon})[(z_a^{\mu} \check{\dot z}_{a'\epsilon}^{\nu} - z_a^{\nu} \check{\dot z}_{a'\epsilon}^{\mu})(\dot z_a \check{\dot z}_{a'\epsilon}) + (z_a^{\mu} \check z_{a'\epsilon}^{\nu} - z_a^{\nu} \check z_{a'\epsilon}^{\mu})(\dot z_a \check\xi_{a'\epsilon})] \\
& - (z_a^{\mu} \check z_{a'\epsilon}^{\nu} - z_a^{\nu} \check z_{a'\epsilon}^{\mu})(\dot z_a \check{\dot z}_{a'\epsilon})[1 + (\check z_{\underline{a}'\epsilon} \check\xi_{a'\epsilon})]\} - \omega^{-1}\gamma_a^{-1} \int_0^{\epsilon\theta} \mathrm{d}\phi |(\check{\dot z}_{a'\epsilon} \check z_{\underline{a}'\epsilon})|^{-3} \\
& \times [\![\{(\check{\dot z}_{a'\epsilon} \check z_{\underline{a}'\epsilon})(\dot z_a \check\xi_{a'\epsilon}) - (\dot z_a \check{\dot z}_{a'\epsilon})[1 + (\check z_{\underline{a}'\epsilon} \check\xi_{a'\epsilon})]\} (z_a^{\mu} \check z_{a'\epsilon}^{\nu} - z_a^{\nu} \check z_{a'\epsilon}^{\mu}) \\
& + (\check{\dot z}_{a'\epsilon} \check z_{\underline{a}'\epsilon})(\dot z_a \check{\dot z}_{a'\epsilon})(z_a^{\mu} \check{\dot z}_{a'\epsilon}^{\nu} - z_a^{\nu} \check{\dot z}_{a'\epsilon}^{\mu})]\!] ;
\end{aligned}
$$

$$A^{\mu}_{a'}(x^{\lambda}_{a}) \equiv \tfrac{1}{2}e_{a'}\sum_{\epsilon}\frac{\hat{u}^{\mu}_{a'\epsilon}}{|(u_{a'\epsilon}\hat{x}_{a'\epsilon})|}.$$

Using the former expressions we have evaluated the energy, momentum and angular momentum for the non-planar solutions of Chern and Havas. After a lengthy calculation, the following results are obtained:

$$\begin{aligned} E \equiv P^0 &= m_a(1-v_a^2)^{1/2}+m_{a'}(1-v_{a'}^2)^{1/2} \\ P^1 &= P^2 = P^3 = 0 \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} L^{03} &= -[m_a h_a(1-v_a^2)^{1/2}+m_{a'}h_{a'}(1-v_{a'}^2)^{1/2}] \\ L^{12} &= -g\frac{1-\chi v_a v_{a'}\cos\theta}{\theta-\chi v_a v_{a'}\sin\theta} \\ L^{01} &= L^{02} = L^{13} = L^{23} = 0. \end{aligned} \tag{4.5}$$

We mention that having conserved total momentum and angular momentum, it is possible to define (Schild 1963) the relativistic centre of mass whose world line c^{α} is the straight line

$$c^{\alpha} = -\frac{L^{\alpha\beta}P_{\beta}}{P^{\mu}P_{\mu}}+\lambda P^{\alpha} \tag{4.6}$$

with λ arbitrary.

Using (4.4), (4.5) and (4.6), we obtain for the coordinates of the centre of mass:

$$c^0 = \lambda P^0, \qquad c^1 = c^2 = 0, \qquad c^3 = (1+S^2)^{-1}h_a+(1+S^{-2})^{-1}h_{a'},$$

$$(S \equiv \gamma_a v_a/\gamma_{a'}v_{a'}).$$

In the centre-of-mass frame, ie in the inertial frame where the world line of the centre of mass of the system coincides with the time axis, so that $c^1 = c^2 = c^3 = 0$, we have that the only surviving components of the conserved quantities are the total energy E and the space component of the angular momentum about the origin L^{12} given by (4.4) and (4.5).

It is important to remark that both solutions ($\chi = +1, \chi = -1$) have the same four-momentum as the planar solutions of Smith and Schild has. We think that this is not an obvious result, but derives from the special form of the interaction function $(\dot{z}_a\dot{z}_{a'})\delta[(z_a-z_{a'})^2]$. More general interaction functions such as other authors consider (Havas 1971, Cordero and Ghirardi 1973), containing Chern and Havas' solutions, will perhaps not exhibit this property (research in this direction will be published elsewhere).

Defining the energy of interaction by

$$E_{\text{int}} = E-m_a\gamma_a-m_{a'}\gamma_{a'}$$

and using (4.4), it follows that

$$E_{\text{int}} = -(m_a\gamma_a v_a^2+m_{a'}\gamma_{a'}v_{a'}^2),$$

that is: the mass defect for these solutions is negative and so we are dealing with bound states. Equation (4.7), together with (2.7) can be interpreted as due to the fact that both particles contribute the same amount to the interaction energy in spite of the fact that they can have different masses.

Regarding the angular momentum, we have shown that in the centre-of-mass frame L^{12} is not the same for $\chi = -1$ and $\chi = +1$, that is: we can distinguish between both solutions. For $\chi = -1$, our results and Schild's (Schild 1963) are equal.

Acknowledgments

We wish to thank one of the referees for his valuable comments.

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